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SOME RESULTS ON INTERPOLATORY BEST APPROXIMATION IN HILBERT SPACES

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Abstract

In this paper, we want to discuss interpolating best approximation in a finite dimensional space and a Hilbert space, and we obtain some properties of this kind.

1. Introduction

Let M be a subset of the normed linear space X. For any $x \in X$, the (possibly empty) set of best approximations to x from M is defined by

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$$P_M(x) = \{y \in M : \|x - y\| = d(x, M)\},$$

where $d(x, M) = \inf\{\|x - y\| : y \in M\}$.

The set M is said to be *proximinal* (resp., *Chebyshev*), if for each $x \in X$, $P_M(x)$ is nonempty (resp., is a singleton). Also, we need to concept of orthogonality. Let X be a normed linear space, if $x, y \in X$, x is said to be *orthogonal* to y and is denoted by $x \perp y$, if and only if $\|x\| \leq \|x + \alpha y\|$ for any scalar α. We called this orthogonality *Birkhoff-orthogonality*. In Hilbert space H for $x, y \in H$, we have $x \perp y$, if and only if $< x, y >= 0$, where $< ., . >$ is inner product in H.

Also, we note that the orthogonality is not symmetric. That is, if $x \perp y$, may be it is not $y \perp x$.

Definition 1.1. Let X be a normed linear space. Orthogonality on X is called *right-additive*, if

$$x \perp z, y \perp z, \text{ then } x + y \perp z,$$

and is called *left-additive*, if

$$z \perp x, z \perp y, \text{ then } z \perp x + y.$$

The product orthogonality in a Hilbert space is right and left additive.

Definition 1.2 [5]**.** Let X be a normed linear space, G be a closed linear subspace of X, $x \in X \setminus G$, and $\{x_1, x_2, \cdots, x_n\}$ be any vector in X.

We call interpolatory of best approximation x with respect to the set $\{x_1, x_2, \cdots, x_n\}$ is an element $g_0 \in G$ with following properties

$$x - g_0 \perp x_i \text{ and } x_i \perp x - g_0,$$

for all $i = 1, 2, \cdots, n$. We put

$$V_x = \{g_0 : x - g_0 \perp x_i \text{ and } x_i \perp x - g_0\}.$$

At first, we have a definition of interpolatory of best approximation.

Lemma 1.3. *Let X be a normed linear space, G be a closed linear subspace of X, $x \in X \setminus G$, and $\{x_1, x_2, \cdots, x_n\}$ be any vector in X. If the orthogonality be a trans-orthogonality. Then*

(1) *V_x is a closed subspace of X.*

(2) *If g_0 is an interpolatory of best approximation x with respect to the set $\{x_1, x_2, \cdots, x_n\}$, then $g_0 \in P_{V_x}(x)$.*

Lemma 1.4. *Let H be a Hilbert space, G be a closed linear subspace of H, $x \in H \setminus G$, and $\{x_1, x_2, \cdots, x_n\}$ be any vector in H. Then, we have*

(1) *We have $V_0 \bigcap g \in G : g \perp x = V_x$. Therefore, g_0 is an interpolatory of best approximation x with respect to $\{x_1, x_2, \cdots, x_n\}$, if $g_0 \perp x$ and $g_0 \perp x_i$ for all i.*

(2) *If g_0 is an interpolatory of best approximation x with respect to the set $\{x_1, x_2, \cdots, x_n\}$, then $< g_0, x - g >= 0$ for all $g \in V_x$.*

(3) *If $g_0 \in P_{V_x}(x)$, then there exists a $y \in H$ such that $\|y\| = 1, < y, g - g_0 >= 0$, and $< y, x - g_0 >= \|x - g_0\|$, where for all $x \in H$, $\|x\|^2 =< x, x >$.*

2. Interpolatory Best Approximation in Finite Dimensional Spaces

Theorem 2.1. *Let X be a finite dimensional space with a basis $\{x_1, x_2, \cdots, x_n\}$. If the orthogonality is right or left additive and $g \perp x_i$ for every $i = 1, 2, \cdots, n$, then $g = 0$.*

Proof. Suppose $g \perp x_i$ for all i. If the orthogonality is right additive, then $g \perp x$ for all $x \in X$. Because $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$. It follows that $\|g\| \leq \|g + x\|$. If we put $x = -g$, then $g = 0$. Also, if orthogonality is left additive, then $\|x\| \leq \|x + g\|$, we put $x = -g$. We have $g = 0$.

Theorem 2.2. *Let X be a finite dimensional space with a basis $\{x_1, x_2, \cdots, x_n\}$. If the orthogonality is right or left additive, then V_x is Chebyshev.*

Proof. Suppose $x \in X$. Then from Lemma 1.4, we show that V_x is proximinal. Suppose $g_1, g_2 \in P_{V_x}(x)$, then $x - g_i \perp x_i$ and $x_i \perp x - g_i$ for every $i = 1, 2$. If orthogonality is right additive $g_1 - g_2 = x - g_2 - (x - g_1) \perp x_i$ for every i, then, by Theorem 2.1, we have $g_1 = g_2$. If the orthogonality is left $x_i \perp x - g_2 - (x - g_1) = g_1 - g_2$ for every i, then, by Theorem 2.1, we have $g_1 = g_2$.

Example 2.3. Suppose $X = R^n$ with the Euclidean norm. Then X is a finite dimensional space. Then, V_x is closed Chebyshev subset of X for every $x \in X$.

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